An introduction to fractal uncertainty principle

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The goal of this minicourse is to give a brief introduction to fractal uncertainty principle and its applications to transfer operators for Schottky groups

Part 1: Schottky groups, transfer operators, and resonances

Schottky groups

Using the action of $SL(2, \mathbb{R})$ on

$$\mathbb{H}^2 = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$$

and on its boundary

$$\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$$

by Möbius transformations:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \gamma . z = \frac{az+b}{cz+d}$$

Schottky groups provide interesting nonlinear dynamics on fractal limit sets and appear in many important applications

To define a Schottky group, we fix:

- a collection of $2\cdot r$ nonintersecting disks in $\mathbb{C}$ with centers in $\mathbb{R}$

  $D_1, \dots, D_{2r}$ $\quad I_j := D_j \cap \mathbb{R}$

- denote $\mathcal{A} := \{1, \dots, 2r\}$ and

$$\bar{a} = \begin{cases} a + r, & \text{if } 1 \leq a \leq r \\ a - r, & \text{if } r < a \leq 2r \end{cases}$$

- fix maps $\gamma_1, \dots, \gamma_{2r}$ such that

$$\gamma_a(\hat{\mathbb{C}} \setminus \mathring{D}_{\bar{a}}) = D_a, \quad \gamma_{\bar{a}} = \gamma_a^{-1}$$

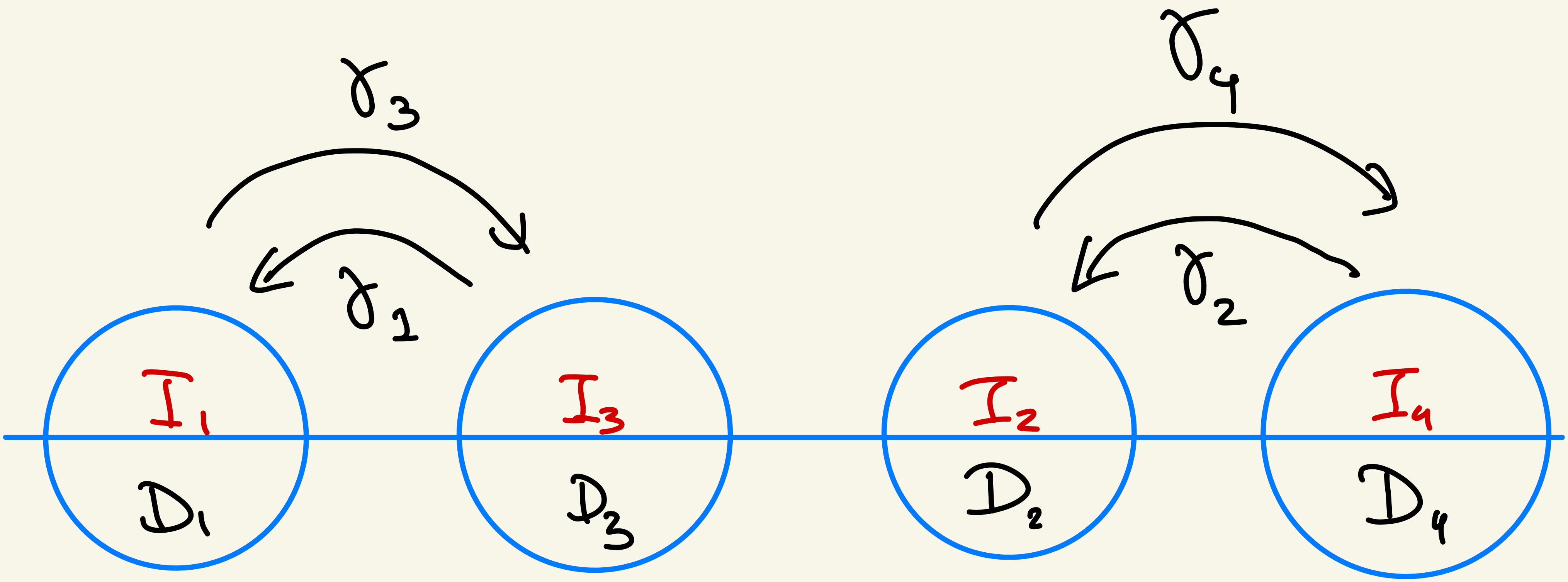
- The Schottky group $\Gamma \subset SL(2,\mathbb{R})$ is the free group generated by $\gamma_1, \dots, \gamma_r$

Example of a Schottky group

Here is a picture for the case of 4 disks:

$$\gamma_1(\dot{\mathbb{C}} \setminus \mathring{D}_3) = D_1, \quad \gamma_2(\dot{\mathbb{C}} \setminus \mathring{D}_4) = D_2,$$

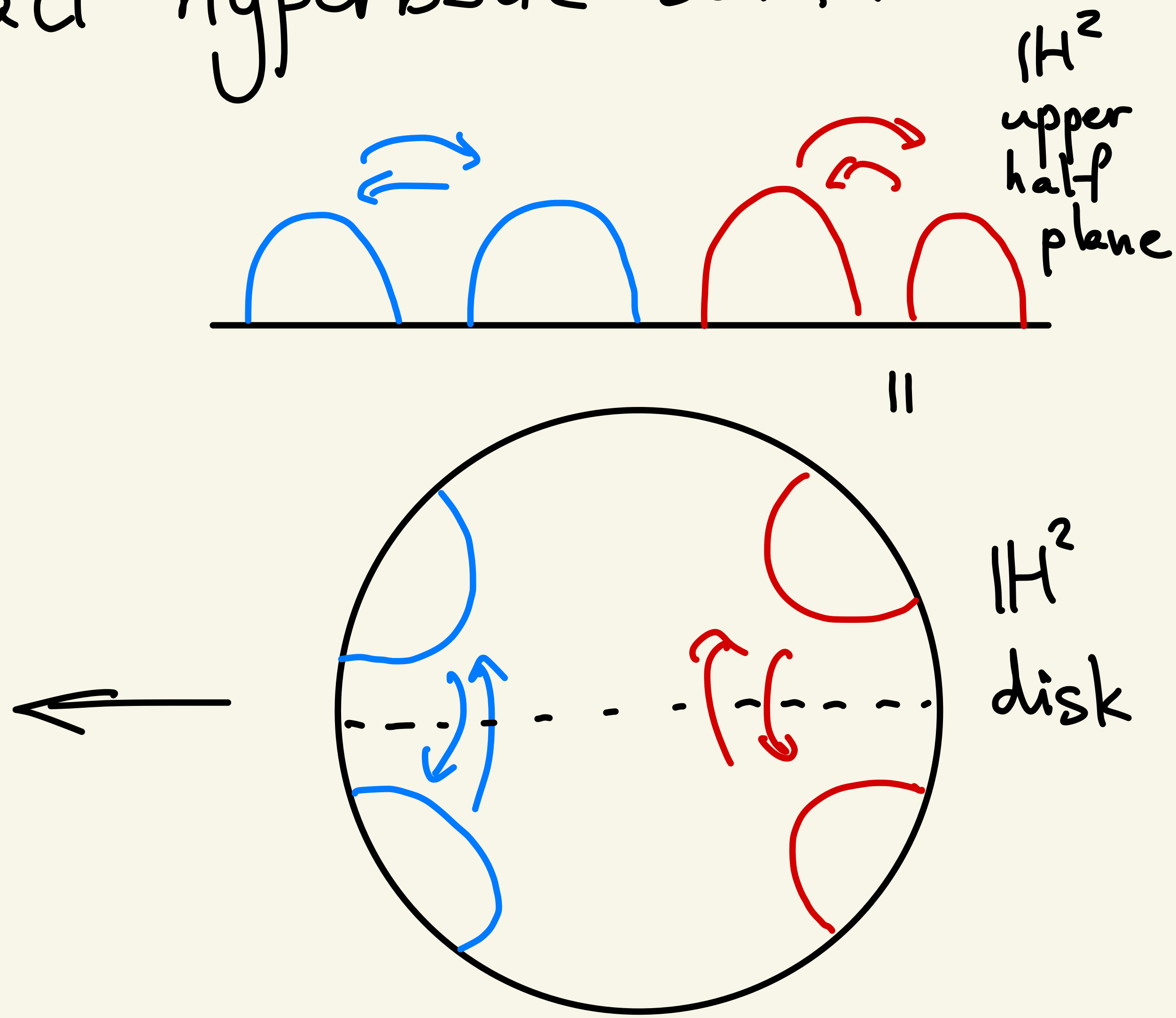
$$\gamma_3(\dot{\mathbb{C}} \setminus \mathring{D}_1) = D_3, \quad \gamma_4(\dot{\mathbb{C}} \setminus \mathring{D}_2) = D_4$$

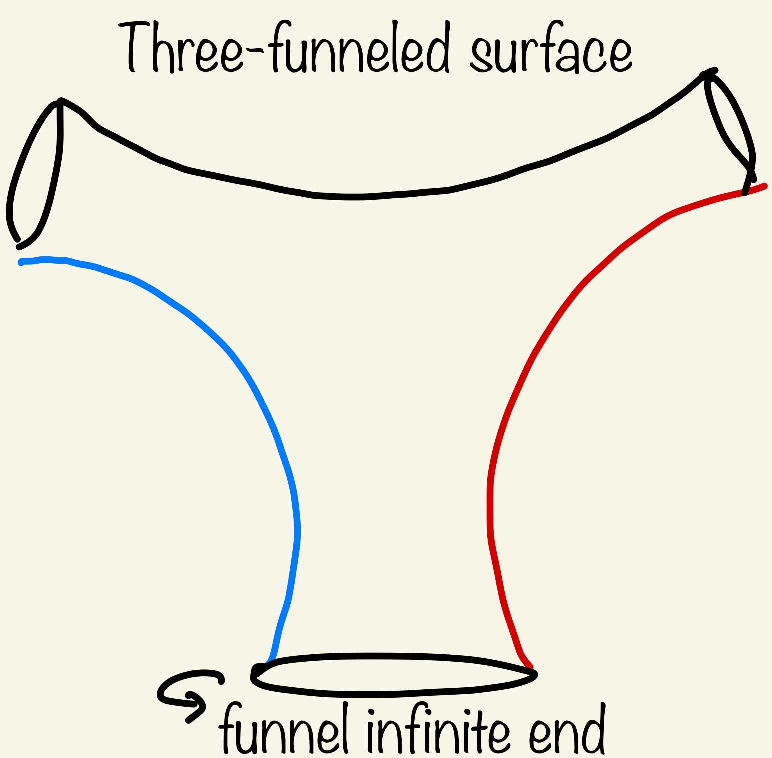


Schottky quotients

Taking the quotient of $\left(\mathbb{H}^2, \frac{|dz|^2}{|\operatorname{Im} z|^2}\right)$ by the action of Γ, we get a convex co-compact hyperbolic surface

$$M = \Gamma \backslash \mathbb{H}^2$$





Words and nested intervals

Recall that $\mathcal{A} = \{1, \dots, 2r\}$ encodes the generators of the group Γ, $\bar{a} := a \pm r$

- Words of length n:

$$W^n = \{a_1 \dots a_n \mid \forall j,\ a_{j+1} \neq \bar{a}_j\}$$

$$\vec{a} = a_1 \dots a_n \implies \vec{a}' := a_1 \dots a_{n-1}$$

- Group elements:

$$\vec{a} = a_1 \dots a_n \longmapsto \gamma_{\vec{a}} := \gamma_{a_1} \cdots \gamma_{a_n} \in \Gamma$$

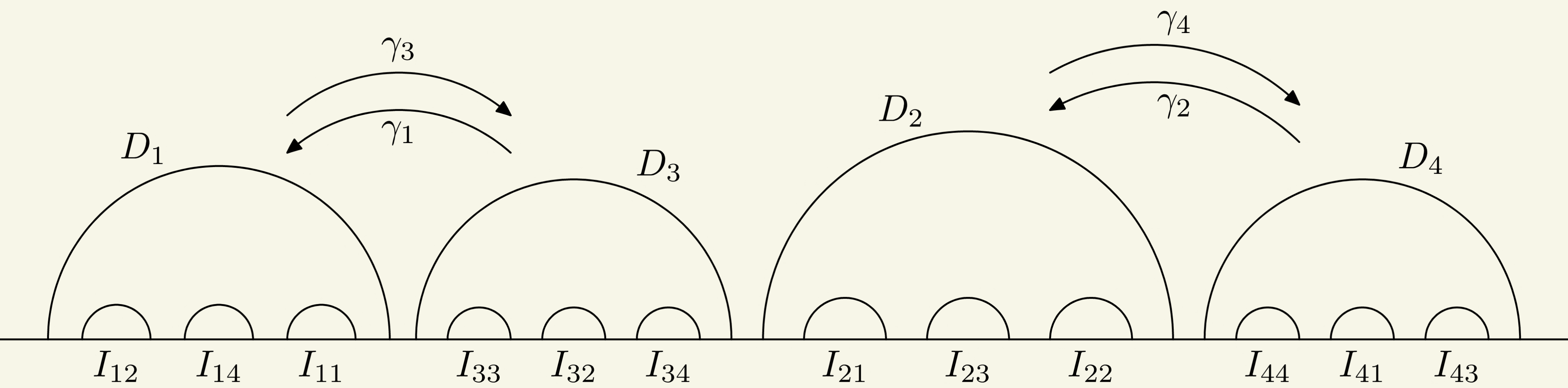
- Intervals / disks:

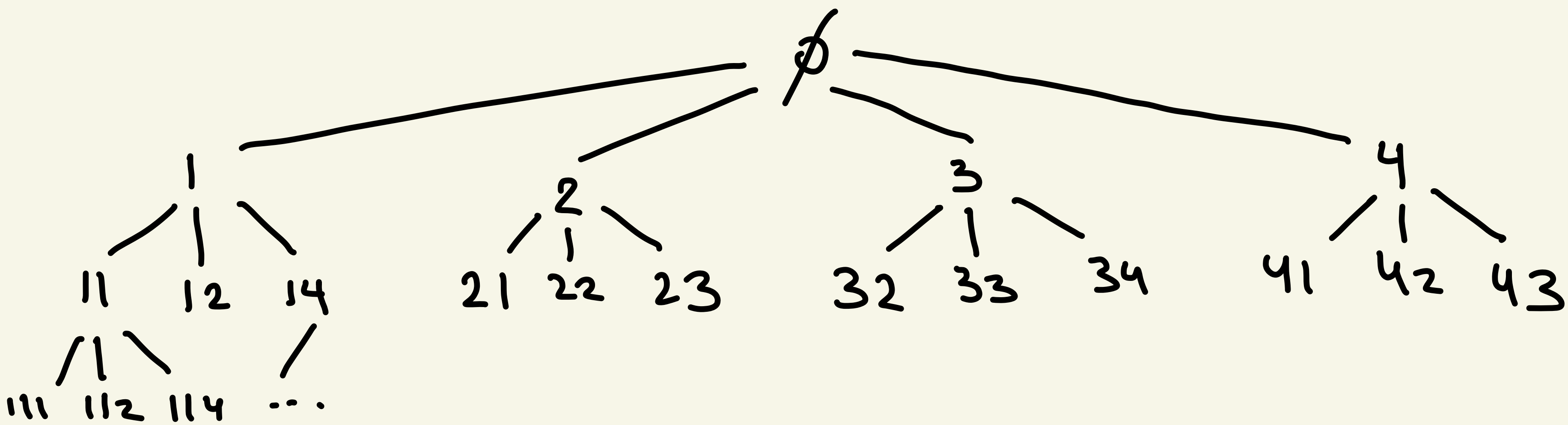
$$D_{\vec{a}} := \gamma_{\vec{a}'}(D_{a_n}), \quad I_{\vec{a}} = \gamma_{\vec{a}'}(I_{a_n})$$

- Nesting property:

$$D_{\vec{a}} \subset D_{\vec{a}'}$$

Picture of the tree of nested disks and intervals





The limit set

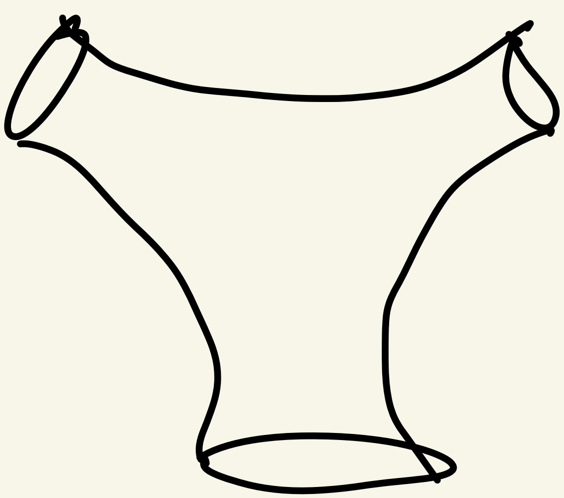
Define the limit set of Γ

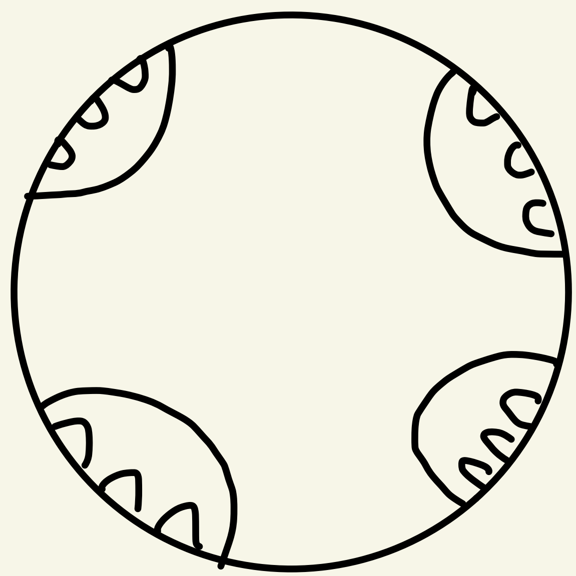
$$\Lambda_\Gamma = \bigcap_{n\geq 1} \bigsqcup_{\vec{a}\in W^n} D_{\vec{a}} \subset \mathbb{R}$$

It is a compact set with fractal structure

Connection to geodesic flow on $M = \Gamma \backslash \mathbb{H}^2$:

A geodesic on M is trapped iff both endpoints of its lift to $\mathbb{H}^2$ lie in Λ_Γ





Transfer operator

Denote by $\mathcal{H}(D)$ the Hilbert space of L^2 holomorphic functions on

$$D := \bigsqcup_{a \in \mathcal{A}} D_a$$

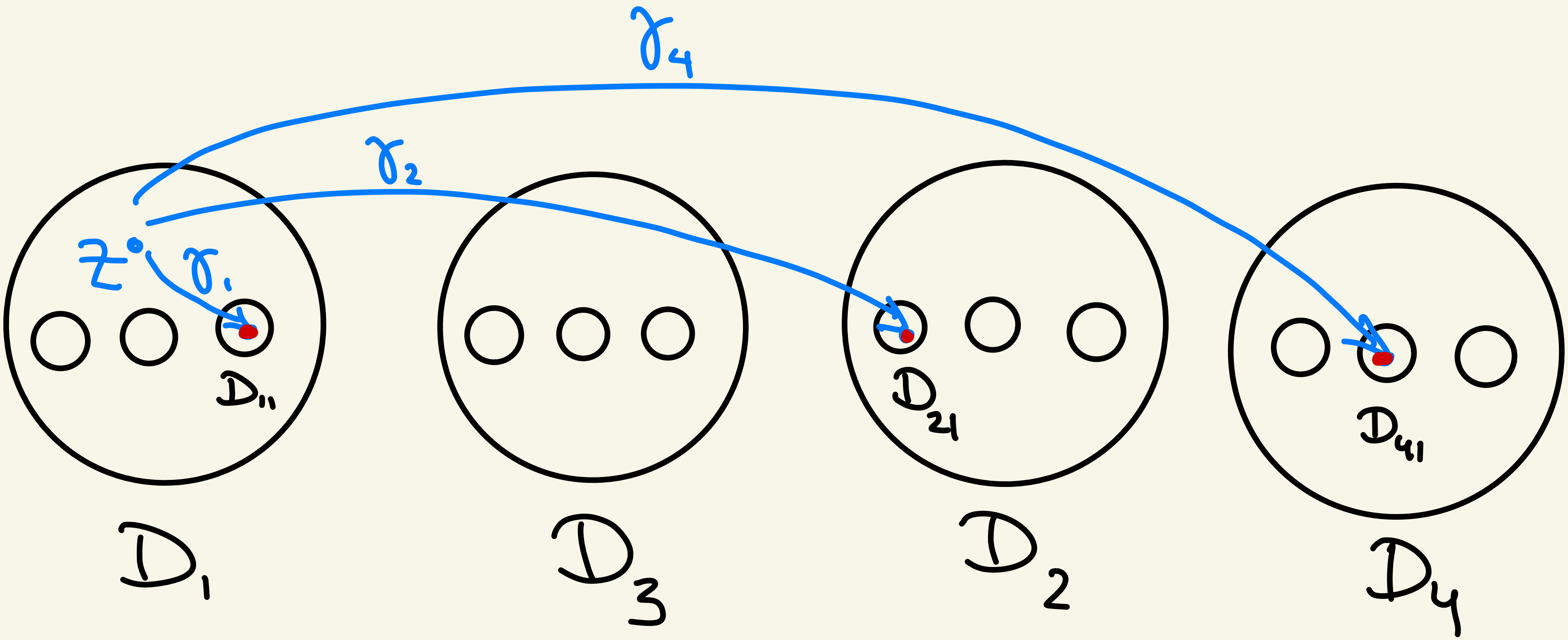
For $s \in \mathbb{C}$, define the <u>transfer operator</u>

$$\mathcal{L}_s : \mathcal{H}(D) \to \mathcal{H}(D)$$

If $f \in \mathcal{H}(D)$ and $z \in D_b$ then

$$\mathcal{L}_s f(z) = \sum_{\substack{a \in \mathcal{A} \\ a \neq \bar{b}}} \left(\gamma_a'(z)\right)^s f(\gamma_a(z))$$

$$z \in D_2$$
$$\Downarrow$$
$$L_s f(z) = \gamma_1'(z)^s f(\gamma_1(z)) + \gamma_2'(z)^s f(\gamma_2(z)) + \gamma_4'(z)^s f(\gamma_4(z))$$



Mapping properties of the transfer operator

$$\mathcal{L}_s f(z) = \sum_{a \neq \bar{b}} \gamma_a'(z)^s f(\gamma_a(z)), \qquad z \in D_b$$

Since $\gamma_a(D_b) \Subset D_a$,

$\mathcal{L}_s : \mathcal{H}(D) \to \mathcal{H}(D)$ is trace class

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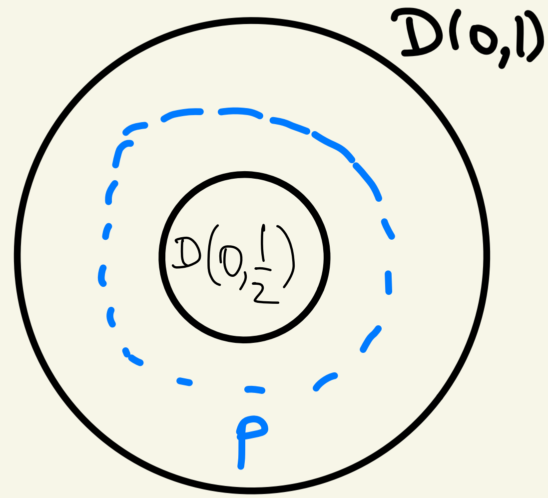
Why trace class? E.g. $\mathcal{L}f(z) = f\left(\frac{z}{2}\right)$

$\mathcal{L} : \mathcal{H}(D(0,1)) \circlearrowleft$, $\quad D(0,1) = \{|z| < 1\}$

$$\mathcal{L}f(z) = \frac{1}{2\pi i} \oint_P \frac{f(t)}{t - \frac{z}{2}} \, dt$$



$\forall t \in P$, $\quad f \mapsto \dfrac{f(t)}{t - z/2}$ is rank 1

The zeta function

Define the Selberg zeta function

$$\zeta(s) := \det(I - \mathcal{L}_s)$$

It can also be expressed in terms of the "set" $\mathcal{L}_M$ of the lengths of primitive closed geodesics on M:

$$\zeta(s) = \prod_{\ell \in \mathcal{L}_M} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell}\right)$$ when $\mathrm{Re}\, s \gg 1$

[Borthwick, Spectral theory of infinite area hyperbolic surfaces]

ζ helps count length spectrum similarly to how the Riemann ζ function helps count primes

Resonances

$$\zeta(s) = \det(I - \mathcal{L}_s) = \prod_{\ell \in \mathcal{L}_M} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell}\right)$$

We call the zeroes of $\zeta(s)$ resonances of M.

Note: s a resonance $\Leftrightarrow$ $I - \mathcal{L}_s$ not invertible $\Leftrightarrow$ $\exists\, u \in \mathcal{H}(D)$: $\mathcal{L}_s u = u$

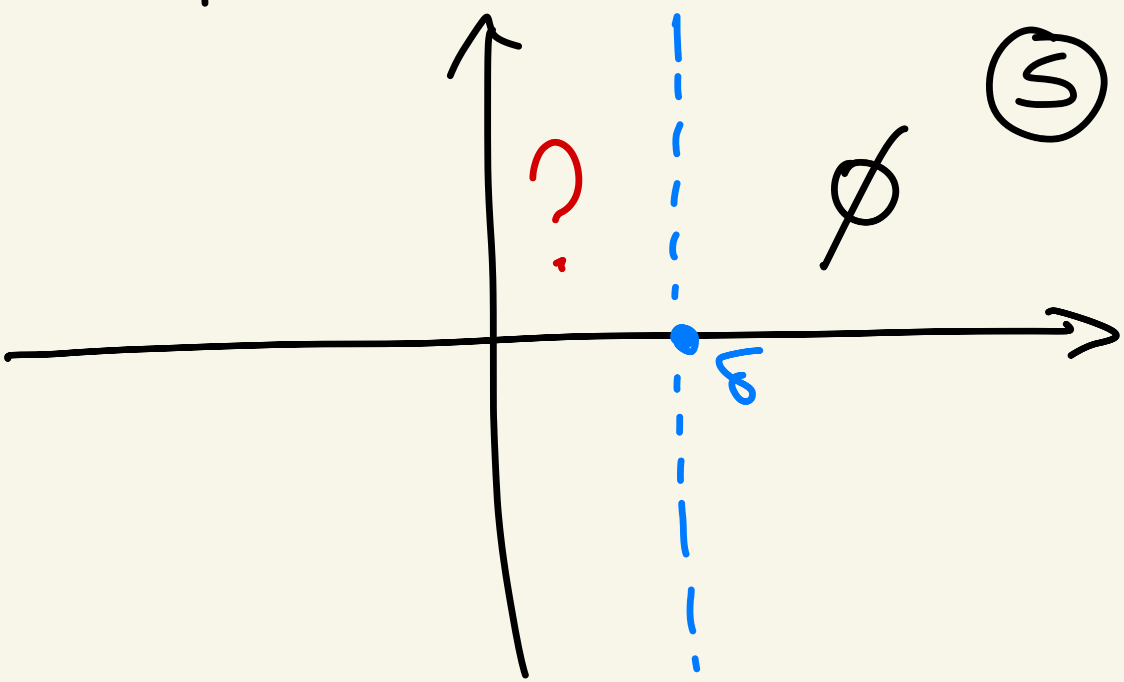
If $\#\{\ell \in \mathcal{L}_M \mid \ell \le T\} \underset{T\to\infty}{=} O(e^{\delta T})$ for some $\delta > 0$, then there are no resonances in $\{\mathrm{Re}\, s > \delta\}$ (the $\prod$ converges)

The converse is true (up to an ε)

Resonance free regions

? What is the smallest δ such that $\zeta(s)$ has no zeroes with $\operatorname{Re} s > \delta$?

! Such δ exists, $0 \le \delta < 1$, δ is a resonance (i.e. $\zeta(\delta)=0$) & there are no other resonances s on the line $\operatorname{Re} s = \delta$

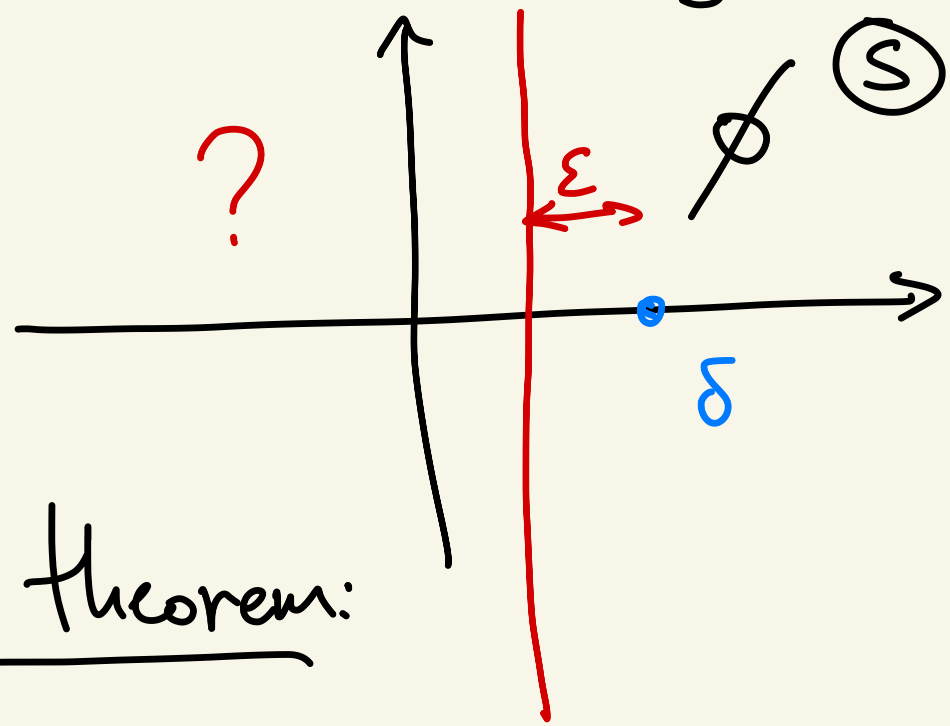


[Patterson, Sullivan]

? Is there $\varepsilon > 0$ such that δ is the only resonance with $\operatorname{Re} s > \delta - \varepsilon$?

! YES, if $\delta > 0$ ($\delta = 0 \to$ 2 disks elementary case)

[Naud 2005, using Dolgopyat 1998]



Application:

exponential remainder in the prime geodesic theorem:

$\exists\ \varepsilon > 0$ (not the same...)

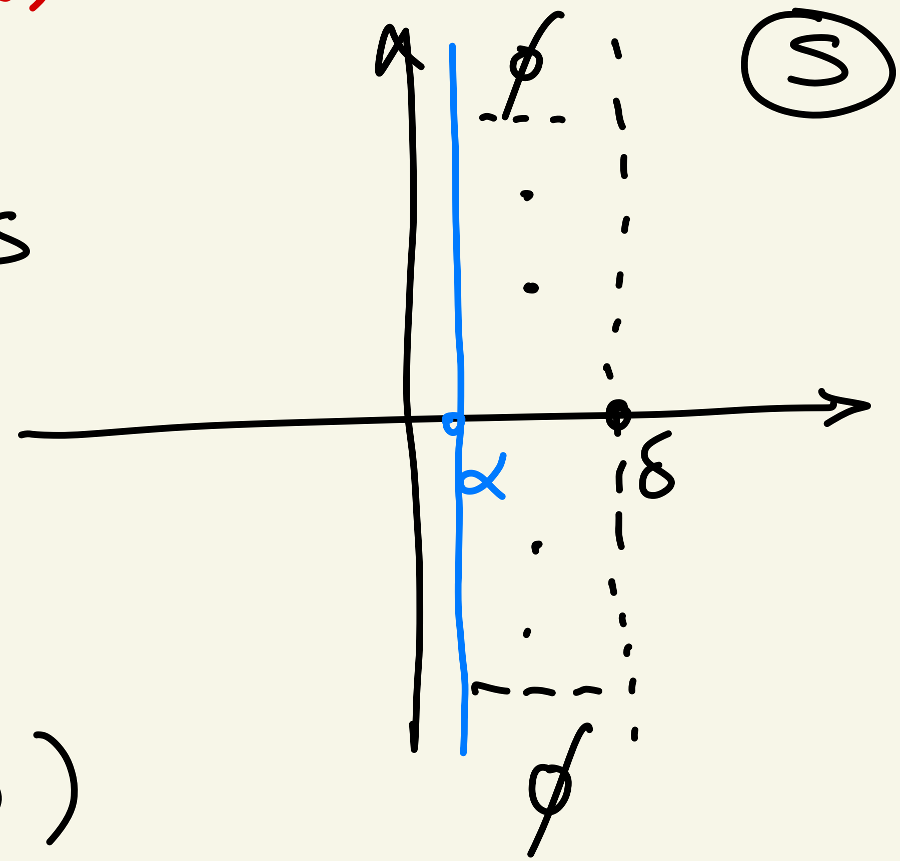
$$\#\{\ell \in \mathcal{L}_M \mid \ell \leq T\} \underset{T\to\infty}{=} \operatorname{li}(e^{\delta T}) + O\left(e^{(\delta-\varepsilon)T}\right)$$

$$\operatorname{li}(x) = \int_2^x \frac{dt}{\ln t} \sim \frac{x}{\ln x}$$

? What is the **smallest α** such that there are only **finitely many** resonances with $\mathrm{Re}\, s > \alpha$?

(SPECTRAL GAP QUESTION)

! WE DON'T KNOW the full answer to this



<u>JAKOBSON–NAUD CONJECTURE:</u>

$\alpha = \delta/2$.

<u>KNOWN:</u>

- $\alpha = \delta - \varepsilon$ (if $\delta > 0$)
- $\alpha = \frac{1}{2}$ [Lax–Phillips] uses spectral theory of Δ_M

Recent results on spectral gaps

$< \infty$ resonances in $\{\mathrm{Re}\, s > \alpha\}$ where

- $\alpha = \frac{1}{2} - \varepsilon, \quad \varepsilon = \varepsilon(\Lambda_\Gamma) > 0$

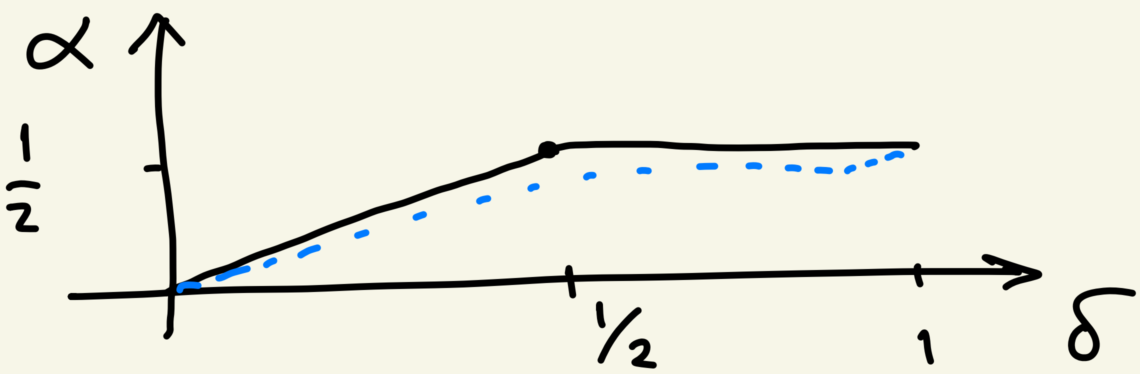
  [Bourgain-D 2018]

- $\alpha = \delta - \varepsilon, \quad \varepsilon = \varepsilon(\delta) > 0$ (when $\delta > 0$)

  [Bourgain-D 2017]

- The above use reduction to fractal uncertainty principle

  [D-Zahl 2016] [D-Zworski 2020]



Gaps for finite covers

Take some family of finite index subgroups $\Gamma_q \subset \Gamma$, then $M_q = \Gamma_q \backslash \mathbb{H}^2$ is a finite cover of $M = \Gamma \backslash \mathbb{H}^2$.

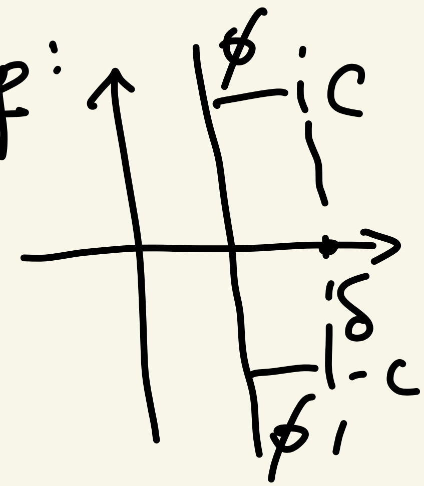
? Is there a uniform spectral gap: $\exists\, \varepsilon > 0\ \forall q$ δ is the only resonance in $\{\operatorname{Re} s > \delta - \varepsilon\}$

! Sometimes yes, sometimes no.

[Bourgain–Gamburd–Sarnak 2011, Oh–Winter 2016, Magee–Oh–Winter 2017, Jakobson–Naud–Soares 2019, Magee–Naud 2019, Magee–Naud–Puder 2020...]

! Always have a high frequency gap: $\exists\, \varepsilon > 0,\ C > 0\ \forall q$ there are no resonances in $\{\operatorname{Re} s > \delta - \varepsilon,\ |\operatorname{Im} s| > C\}$

[Magee–Naud 2019]



Patterson–Sullivan measure

The P-S measure is a probability measure μ on the limit set Λ_Γ which is Γ-equivariant:

$$\int_{\Lambda_\Gamma} f \, d\mu = \int_{\Lambda_\Gamma} f(\gamma(x)) (\gamma'(x))^\delta \, d\mu(x) \quad \forall \gamma \in \Gamma$$

If $\mathcal{L}_\delta$ is the transfer operator

$$\mathcal{L}_\delta f(x) = \sum_{a \neq \bar{b}} \gamma_a'(x)^\delta f(\gamma_a(x)), \quad x \in I_b$$

then μ spans the kernel of $I - \mathcal{L}_\delta^*$:

$$\forall f, \quad \int_{\Lambda_\Gamma} f \, d\mu = \int_{\Lambda_\Gamma} (\mathcal{L}_\delta f) \, d\mu$$

Regularity of the Patterson–Sullivan measure

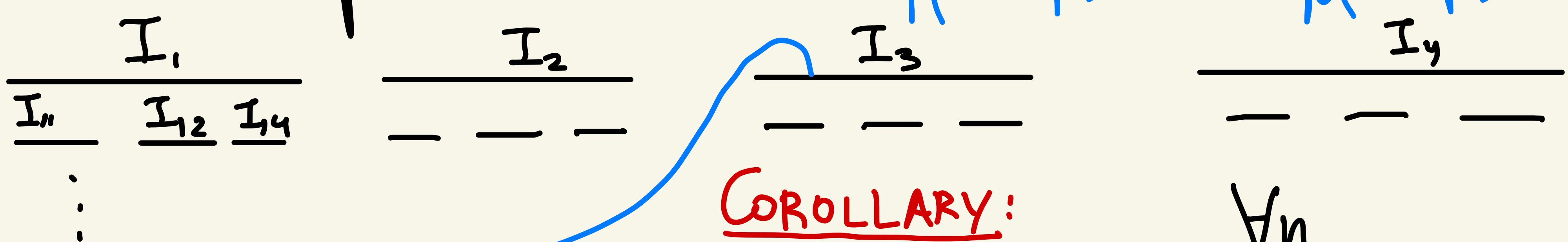
$C^{-1} \le \frac{(\dots)}{(\dots)} \le C$

Here are some basic properties of Schottky groups:

- If $\vec{a} = a_1 \dots a_n \in \mathcal{W}^n$ then $\gamma'_{\vec{a}'}(x) \sim |I_{\vec{a}}|$
  here $\vec{a}' = a_1 \dots a_{n-1}$, $I_a = \gamma_{\vec{a}'}(I_{a_n})$
- $\mu(I_{\vec{a}}) = \mu(\gamma_{\vec{a}'}(I_{a_n})) = \int_{I_{a_n}} |\gamma_{\vec{a}'}(x)|^{\delta} \, d\mu(x)$

Therefore $\mu(I_{\vec{a}}) \sim |I_{\vec{a}}|^{\delta}$

This is called δ-regularity of μ and implies that $\dim_H(\Lambda_\Gamma) = \dim_M(\Lambda_\Gamma) = \delta$



$I_{\vec{a}}$ $\qquad \gamma_{\vec{a}'}$

COROLLARY: $\forall n$,

$$\sum_{\vec{a} \in \mathcal{W}^n} |I_{\vec{a}}|^{\delta} \sim \sum_{\vec{a} \in \mathcal{W}^n} \mu(I_{\vec{a}}) = 1$$

Part 2: from fractal uncertainty principle to spectral gap

The standard gap

Recall: $\mathcal{L}_s f(z) = \sum_{a \neq b} \gamma_a'(z)^s f(\gamma_a(z)), \quad z \in D_b$

THEOREM If $\operatorname{Re} s > \delta$ then $\det(I - \mathcal{L}_s) \neq 0$

PROOF Assume not. Then $\exists u \in \mathcal{H}(D)$: $\mathcal{L}_s u = u$

Thus $\forall n$, $\mathcal{L}_s^n u = u$

Now $\mathcal{L}_s^n u(z) = \sum_{\substack{\vec{a} = a_1 \cdots a_n \\ a_n \neq b}} \gamma_{\vec{a}}'(z)^s u(\gamma_{\vec{a}}(z)), \quad z \in D_b$

So $\sup_{\Lambda_\Gamma} |\mathcal{L}_s^n u| \leq (\sup_{\Lambda_\Gamma} |u|) \cdot \sum_{\vec{a} \in W_n} |I_{\vec{a}}|^s$.

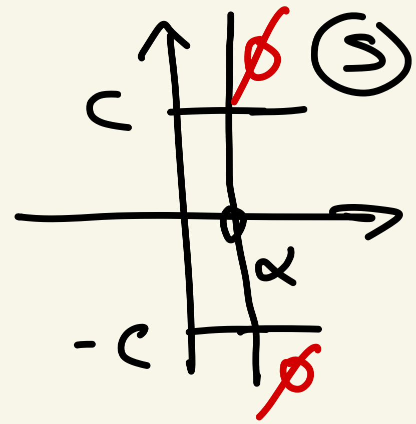
But $\sum_{\vec{a} \in W^n} |I_{\vec{a}}|^\delta \sim 1$, $\operatorname{Re} s > \delta$, and

$\max_{\vec{a} \in W^n} |I_{\vec{a}}| \xrightarrow[n \to \infty]{} 0$. So $\sum_{\vec{a} \in W^n} |I_{\vec{a}}|^s \xrightarrow{n \to \infty} 0$ and thus $u = 0$. $\square$

Improving over the standard gap

- We want to show there are only finitely many resonances with $\operatorname{Re} s > \alpha$, for some $\alpha < \delta$
- Since resonances form a discrete set, this is equivalent to the high frequency statement:

$\exists C > 0$: NO RESONANCES s WITH $\operatorname{Re} s > \alpha$, $|\operatorname{Im} s| > C$



- Assume s is a resonance.

Then $\exists\, u \in \mathcal{H}(D)$: $\mathcal{L}_s u = u$

This implies $\mathcal{L}_s^n u = u$ for all n

• Take $x \in I_b \subset \mathbb{R}$. Then

$$u(x) = \mathcal{L}_s^n u(x) = \sum_{\vec{a} = a_1 \dots a_n \in W_n,\ \bar{a}_n \neq b} \gamma'_{\vec{a}}(x)^s \, u(\gamma_{\vec{a}}(x))$$

Write $\boxed{s = \sigma + i/h}$ where $\sigma > \alpha$, $0 < h \ll 1$

Then $u(x) = \sum_{\vec{a} \in W_{n,\dots}} \gamma'_{\vec{a}}(x)^\sigma e^{\frac{i}{h} \log \gamma'_{\vec{a}}(x)} u(\gamma_{\vec{a}}(x))$

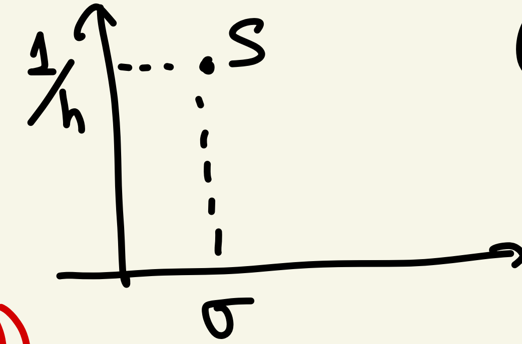
• $\gamma'_{\vec{a}}(x)^\sigma > 0$. If $\sigma < \delta$ then $\sum_{\vec{a}} \gamma'_{\vec{a}}(x)^\sigma \gg 1$

• $e^{\frac{i}{h} \log \gamma'_{\vec{a}}(x)}$ oscillates at frequency $\sim 1/h$ (wavelength h)

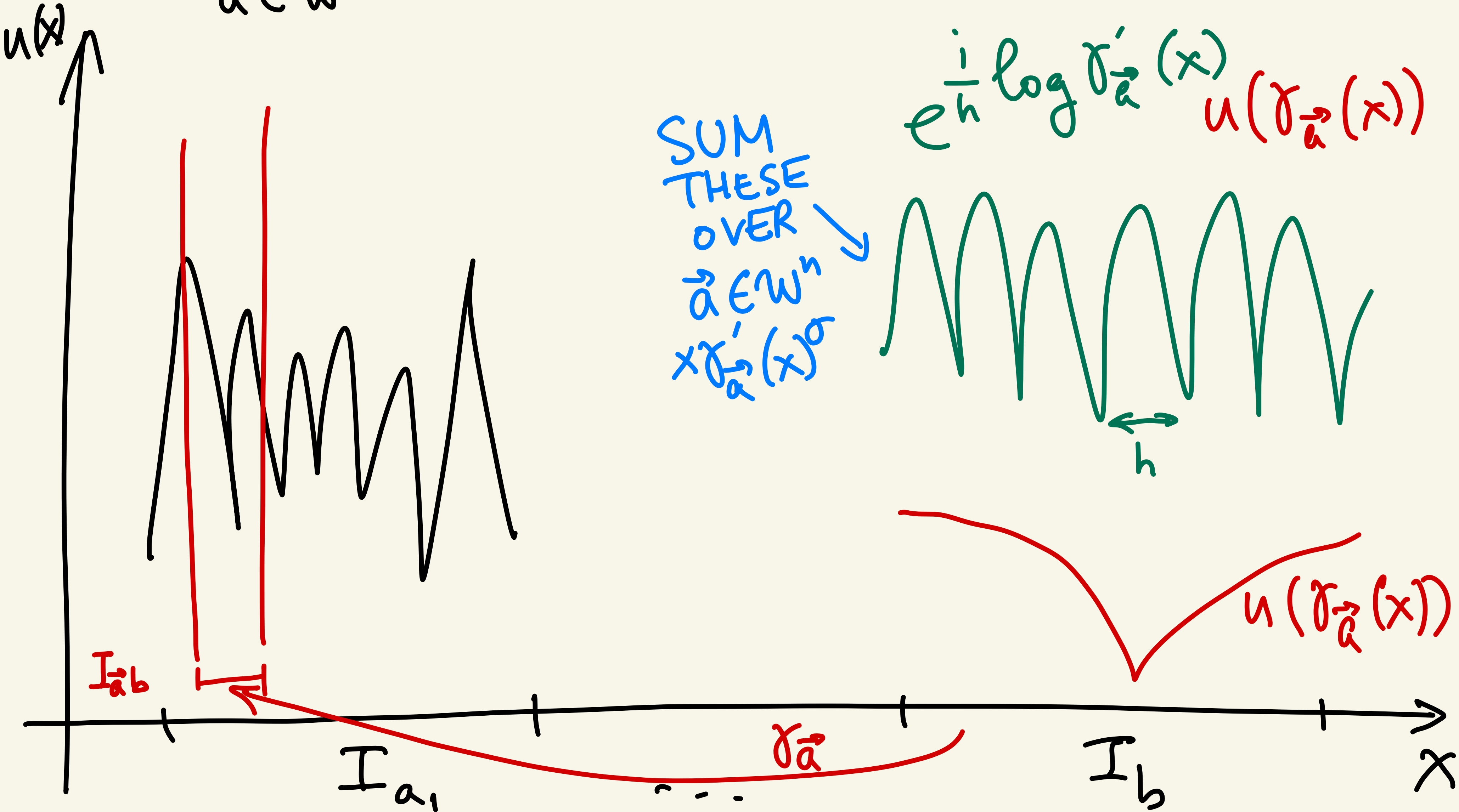
• $u \mapsto u(\gamma_{\vec{a}}(x))$ smoothens out: reduces frequency by $|\gamma'_{\vec{a}}| \sim |I_{\vec{a}}|$

How fast does u oscillate?

$u \in \mathcal{H}(D)$, $u = \mathcal{L}_s u = \mathcal{L}_s^n u$,



$$u(x) = \sum_{\vec{a} \in W^n} \gamma'_{\vec{a}}(x)^{\sigma} e^{\frac{i}{h} \log \gamma'_{\vec{a}}(x)} u(\gamma_{\vec{a}}(x))$$



CONCLUSION: We expect that u oscillates at frequency $\sim 1/h$ i.e. at wave length $\sim h$.

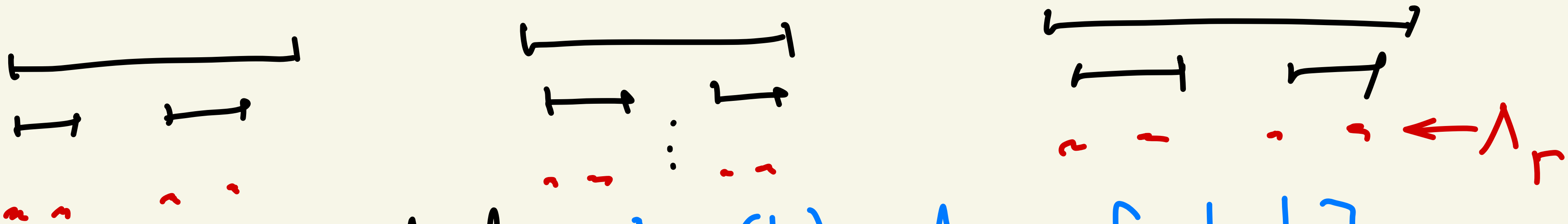
(!) The factors $e^{\frac{i}{h}\log \sigma'_{\vec{a}}(x)}$ oscillate at different frequencies for different $\vec{a}$.

So when $h \ll 1$ we can hope to exploit cancellations in $\sum_{\vec{a} \in W^n}$ to get decay of $\sum$ (and thus $u = 0$) even when $\sigma = \mathrm{Re}\, s < \delta$

This is very roughly how the method of Dolgopyat works...

Fractal uncertainty principle

- In the sum above, $u(\gamma_{\vec{a}}(x))$ only depends on $u|_{I_{\vec{a}b}}$. For large n, $I_{\vec{a}b}$ is close to the limit set Λ_Γ:



- For $h>0$, let $\Lambda_\Gamma(h) = \Lambda_\Gamma + [-h,h]$ be the h-fattening of Λ_Γ
- For $\chi \in C_0^\infty(\mathbb{R}^2)$, $\operatorname{supp}\chi \cap \{x=y\} = \emptyset$ define the operator $B_\chi(h) : L^2(\mathbb{R}) \circlearrowleft$

$$B_\chi(h) f(x) = (2\pi h)^{-\frac{1}{2}} \int_{\mathbb{R}} |x-y|^{-\frac{2i}{h}} \chi(x,y) f(y)\,dy$$

DEFINITION We say Λ_Γ Satisfies the FRACTAL UNCERTAINTY PRINCIPLE with exponent β, if $\forall \chi$, as $h \to 0$

$$\| \mathbb{1}_{\Lambda_\Gamma(h)} B_\chi(h) \mathbb{1}_{\Lambda_\Gamma(h)} \|_{L^2(\mathbb{R}) \circlearrowleft} = O(h^\beta)$$

That is:

if $f \in L^2(\mathbb{R})$ and $\operatorname{supp} f \subset \Lambda_\Gamma(h)$

then $\| B_\chi(h) f \|_{L^2(\Lambda_\Gamma(h))} \leq C h^\beta \| f \|_{L^2}$.

WHY "UNCERTAINTY PRINCIPLE"?

$\operatorname{supp} f \subset \Lambda_\Gamma(h) \Rightarrow v := B_\chi(h) f$ is localized in frequency

$\| v \|_{L^2(\Lambda_\Gamma(h))}$ localizes v in position

A more basic form of FUP replaces $\mathcal{B}_\chi(h)$ by the Fourier transform

$$\mathcal{F}_h f(x) = (2\pi h)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{i}{h} x\cdot y} f(y)\, dy$$

$$\|\mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y\|_{L^2 \to} = O(h^\beta) \iff \forall v \in L^2(\mathbb{R}) \text{ if } \operatorname{supp} \hat{v} \subset h^{-1}\cdot Y \text{ then } \|v\|_{L^2(X)} \leq C h^\beta \|v\|_{L^2(\mathbb{R})}$$

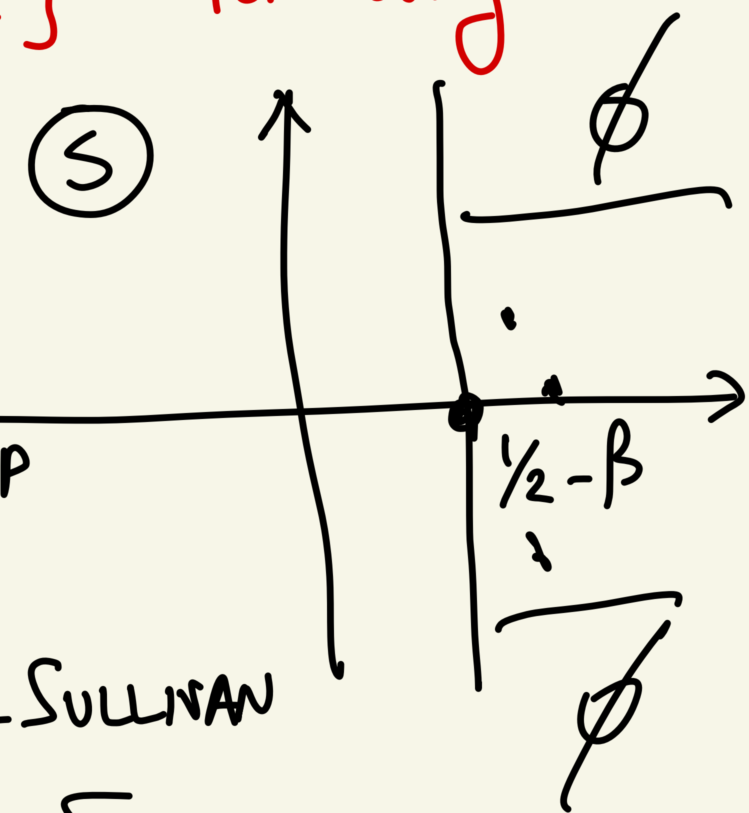
- $\|\mathcal{B}_\chi(h)\|_{L^2 \to} \leq C \Rightarrow$ FUP holds with $\beta = 0$
- $\|\mathcal{B}_\chi(h)\|_{L^1 \to L^\infty} = O(h^{-\frac{1}{2}})$, $|\Lambda_\Gamma(h)| \sim h^{1-\delta} \Rightarrow$

$$\Rightarrow \|\mathbb{1}_{\Lambda_\Gamma(h)} \mathcal{B}_\chi(h) \mathbb{1}_{\Lambda_\Gamma(h)}\|_{L^2 \to} \leq \sqrt{|\Lambda_\Gamma(h)|} \cdot C h^{-\frac{1}{2}} \cdot \sqrt{|\Lambda_\Gamma(h)|}$$

$\Rightarrow$ FUP holds with $\beta = \frac{1}{2} - \delta$

THEOREM Assume Λ_Γ satisfies FUP with exponent β. Then $M = \Gamma \backslash \mathbb{H}^2$ has only finitely many resonances in $\{\operatorname{Re} s > \frac{1}{2} - \beta + \epsilon\}$ for any $\epsilon > 0$

[D-Zahl 2016, D-Zworski 2020]



Note: up to an ϵ,

FUP with $\beta = 0 \Rightarrow$ LAX-PHILLIPS GAP $\operatorname{Re} s > \frac{1}{2}$

FUP with $\beta = \delta \Rightarrow$ PATTERSON-SULLIVAN GAP $\operatorname{Re} s > \delta$

Proof of Theorem (FUP implies spectral gap)

1. Setup

We need $\det(I - \mathcal{L}_s) \neq 0$.

Assume the contrary, then $\exists u \in \mathcal{H}(D) : \mathcal{L}_s u = u$

where $s = \sigma + \frac{i}{h}$, $\sigma > \frac{1}{2} - \beta + \epsilon$, $0 < h \ll 1$

We have $u = \mathcal{L}_s^n u$, i.e. for $z \in D_b$

$$u(z) = \sum_{\vec{a} \in W^n,\, a_n \neq b} \gamma_{\vec{a}}'(z)^s \, u(\gamma_{\vec{a}}(z))$$

CHOOSE n so that

$|I_{\vec{a}}| \sim h$ for all $\vec{a} \in W^n$

(Not really possible. In reality $\mathcal{L}_s^n$ is replaced by an "adapted power" of $\mathcal{L}_s$)

2. Rough localization in frequency

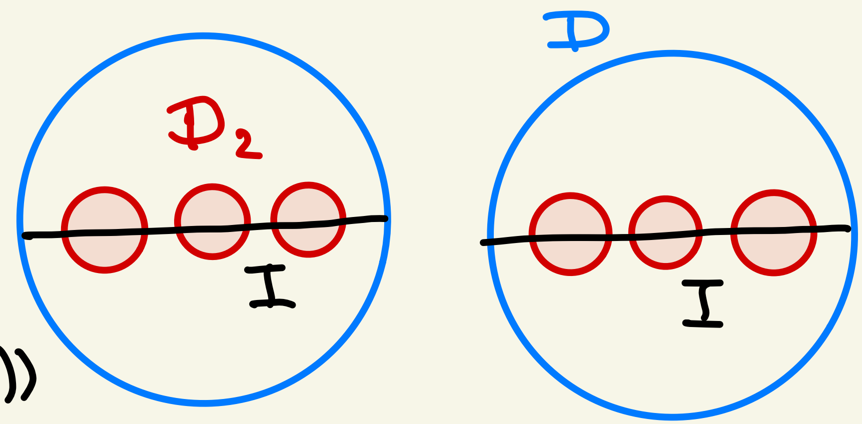
Claim: $u|_{\mathbb{R}}$ lives at frequencies $\lesssim 1/h$, i.e.

for $|\xi| \geq C$, $|\hat{u}(\xi/h)| = O(h^\infty |\xi|^{-\infty})$

Proof: put $\mathcal{D} = \bigsqcup_{a \in \mathcal{A}} \mathcal{D}_a \supset \mathcal{D}_2 = \bigsqcup_{\vec{a} \in W^2} \mathcal{D}_{\vec{a}}$

Define the weight

$$\boxed{w(z) = \exp\left(-\frac{K|\operatorname{Im} z|}{h}\right)}$$



Since $u(z) = \sum_{a \in \mathcal{A}, \dots} \gamma_a'(z)^s u(\gamma_a(z))$

and $\gamma_a(z) \in \mathcal{D}_2$, for $K \gg 1$ we get

$$\boxed{\sup_{\mathcal{D}} |w_K \cdot u| \leq C \sup_{\mathcal{D}_2} |w_K \cdot u| \leq C \left(\sup_I |u|\right)^\alpha \cdot \left(\sup_{\mathcal{D}} |w_K \cdot u|\right)^{1-\alpha}}$$

So $\sup_{\mathcal{D}} |w_K \cdot u| \leq C \sup_I |u|$ which implies the claim.

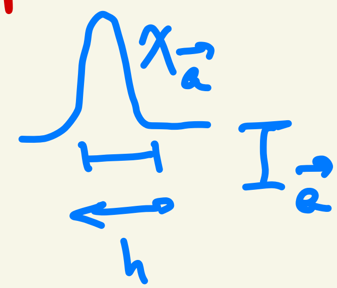
3. Cutting into pieces

From now on we only study $u|_{\mathbb{R}}$. Recall:

$$(*) \quad u(x) = \sum_{\vec{a} \in W^n \dots} \gamma'_{\vec{a}}(x)^s \, u(\gamma_{\vec{a}}(x))$$

this only depends on $u|_{I_{\vec{a}}}$

Define $u_{\vec{a}} = \chi_{\vec{a}} \cdot u \in C_c^\infty(I_{\vec{a}})$



Note: $u_{\vec{a}}$ is still localized at frequencies $\lesssim h^{-1}$ ($\chi_{\vec{a}}$ does not spoil this)

Recall the operator featured in FUP

$$\mathcal{B}_\chi(h) f(x) = (2\pi h)^{-\frac{1}{2}} \int_{\mathbb{R}} |x-y|^{-\frac{2i}{h}} \chi(x,y) f(y)\, dy$$

We use a closely related operator

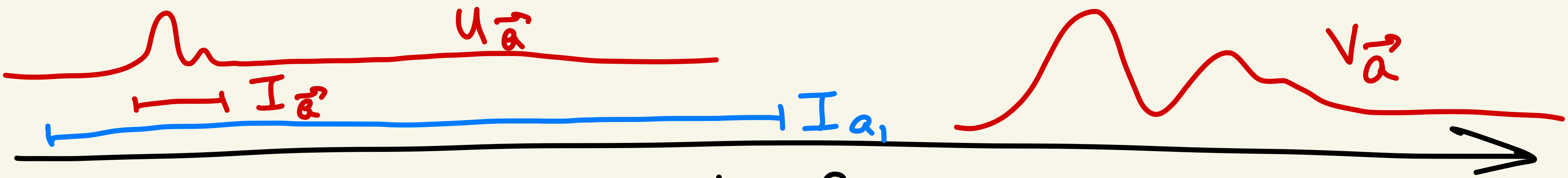
$$\mathcal{B} f(x) = (2\pi h)^{-\frac{1}{2}} \int_{\mathbb{R}} |x-y|^{-2s} f(y)\, dy$$

(recall $s = \sigma + i/h$)

<u>Claim:</u> we can write

$$u_{\vec{a}} = \chi_{\vec{a}} \mathcal{B} v_{\vec{a}} + O(h^\infty) \text{ for some}$$

$v_{\vec{a}}$, $\|v_{\vec{a}}\|_{L^2(\mathbb{R})} \leq C \|u_{\vec{a}}\|_{L^2(\mathbb{R})}$, $\operatorname{supp} v_{\vec{a}} \cap I_{a_1} = \emptyset$



The proof uses a bit of microlocal analysis...

- $\mathcal{B}$ is "unitary", similar to Fourier transform

$$\mathcal{F}_h f(x) = (2\pi h)^{-\frac{1}{2}} \int e^{-\frac{i}{h}xy} f(y)\, dy$$

So put "$v_{\vec{a}} := \mathcal{B}^{-1} u_{\vec{a}}$"

- The fact that $\operatorname{supp} v_{\vec{a}}$ is far from $I_{\vec{a}}$ follows from $u_{\vec{a}}$ being localized to frequencies $\lesssim \frac{1}{h}$:

if $\operatorname{supp} \vec{v}_a$ is close to $I_{\vec{a}}$ then $u_{\vec{a}} = \mathcal{B} v_{\vec{a}}$ oscillates too fast

4. Manipulating the sum

By $(*)$, $u(x) = \sum_{\vec{a} \in W^n \dots} \gamma'_{\vec{a}}(x)^s u_{\vec{a}}(\gamma_{\vec{a}}(x))$

$$= \sum_{\vec{a} \in W^n \dots} \chi_{\vec{a}}(\gamma_{\vec{a}}(x))\, \gamma'_{\vec{a}}(x)^s \, (\mathcal{B} v_{\vec{a}})(\gamma_{\vec{a}}(x))$$

From the definition $\boxed{\mathcal{B} f(x) = (2\pi h)^{-\frac{1}{2}} \int |x-y|^{-2s} f(y)\,dy}$

we get an equivariance property:

$$\gamma'_{\vec{a}}(x)^s (\mathcal{B} \vec{v}_a)(\gamma_{\vec{a}}(x)) = \mathcal{B}\left((\gamma'_{\vec{a}})^{1-s} \cdot (\vec{v}_{\vec{a}} \circ \gamma_{\vec{a}})\right)(x)$$

To show this property we use the relation

$$|\gamma(x) - \gamma(y)|^2 = |x-y|^2 \cdot \gamma'(x) \cdot \gamma'(y)$$

which is where the choice of $|x-y|$ in $\mathcal{B}$ becomes important

Denote $W_{\vec{a}}(x) = \gamma'_{\vec{a}}(x)^{1-s} \, v_{\vec{a}}(\gamma_{\vec{a}}(x))$

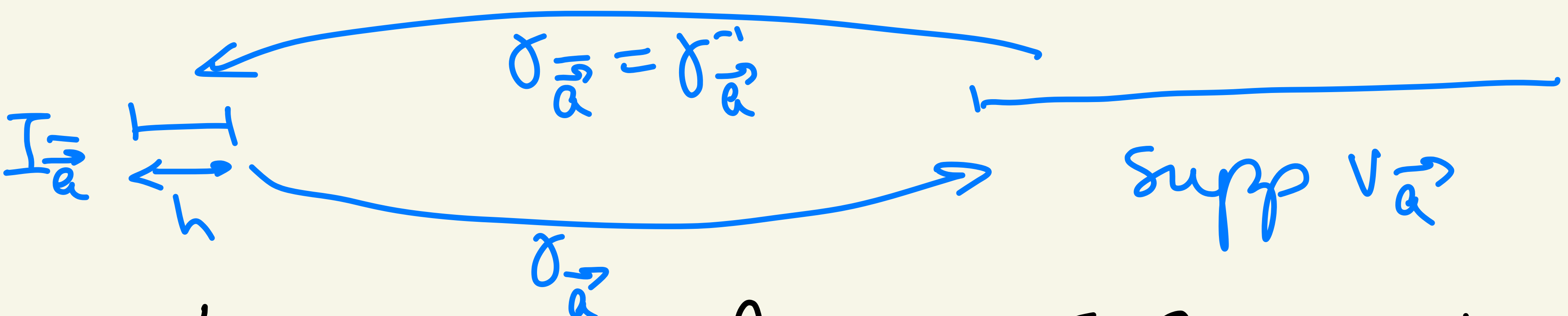
Then $(*)$ gives (up to $O(h^\infty)$)

$$u(x) = \sum_{\vec{a} \in W^n \dots} \chi_{\vec{a}}(\gamma_{\vec{a}}(x)) \, \mathcal{B} \, w_{\vec{a}}(x).$$

Properties of $w_{\vec{a}}$:

- supp $w_{\vec{a}} = \gamma_{\vec{a}}^{-1}(\text{supp } v_{\vec{a}}) \subset I_{\bar{\vec{a}}}$ since supp $v_a \cap I_{a_1} = \emptyset$

  Here $\bar{\vec{a}} = \bar{a}_n \dots \bar{a}_1$ where $\vec{a} = a_1 \dots a_n$



- $\gamma'_{\vec{a}}(x) \sim h^{-1}$ for $x \in$ supp $w_{\vec{a}}$, so

$$\|w_{\vec{a}}\|_{L^2} \sim h^{\mathrm{Re}\, s - \frac{1}{2}} \|v_{\vec{a}}\|_{L^2} \sim h^{\sigma - \frac{1}{2}} \|u_{\vec{a}}\|_{L^2}$$

(recall that $s = \sigma + i/h$)

5. Using FUP to finish the proof

Denote $U = \sum_{\vec{a} \in \mathcal{W}^n} u_{\vec{a}}$, $W = \sum_{\vec{a} \in \mathcal{W}^n} w_{\vec{a}}$.

Then $(*)$ gives (up to $O(h^\infty)$)

$$\boxed{U = \mathcal{X} B W = \mathcal{X} B \mathcal{X} W}$$

where

$$\mathcal{X} = \sum_{\vec{a} \in \mathcal{W}^n} \mathcal{X}_{\vec{a}} \sim \mathbb{1}_{\Lambda_\Gamma(h)}, \quad B \sim \mathcal{B}_\chi(h)$$

And $\|U\|^2_{L^2} \sim \sum_{\vec{a}} \|u_{\vec{a}}\|^2$, $\|W\|^2_{L^2} \sim \sum_{\vec{a}} \|w_{\vec{a}}\|^2$,

$\|w_{\vec{a}}\|_{L^2} \sim h^{\sigma - \frac{1}{2}} \|u_{\vec{a}}\|_{L^2}$. Thus

$$\|U\|_{L^2} \lesssim \|\mathbb{1}_{\Lambda_\Gamma(h)} \mathcal{B}_\chi(h) \mathbb{1}_{\Lambda_\Gamma(h)}\|_{L^2 \to L^2} \cdot h^{\sigma - \frac{1}{2}} \|U\|_{L^2}$$

$$\overset{\text{FUP}}{\lesssim} h^{\beta + \sigma - \frac{1}{2}} \|U\|_{L^2}.$$

Since $\sigma > \frac{1}{2} - \beta + \epsilon$ we get $U = 0$. $\square$

Part 3: Fourier decay and fractal uncertainty principle

TODO